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The Kelvin–Helmholtz instability of the gas-liquid interface of a sonic gas jet submerged in a liquid

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It is well known that the small perturbation equation governing steady or mildly unsteady potential flow in a sonic gas jet is nonlinear. However, for a sonic gas jet submerged in a liquid with a disturbance on the gas–liquid interface, it is shown that the transient motion of the gas dominates, and the nonlinear term due to accumulation of disturbances in the basic flow becomes negligible; the condition necessary for the applicability of the linearized governing equation is obtained. It is demonstrated that most gas-jet/liquid systems of physical interest satisfy this condition and that the margin with which this condition is satisfied improves as the wave velocity of the disturbance or, more particularly, as the stagnation pressure or density of the gas for a given gas–liquid system increases. The Kelvin–Helmholtz instability of the gas–liquid interface of a sonic gas jet submerged in a liquid is predominantly governed by the transfer of energy from the gas phase to the liquid layer, both through wave-drag and 'lift' components of the pressure perturbation; at and above the cut-off wavenumber, which only exists for very low viscosity liquids owing to the stabilizing effect of surface tension, the pressure perturbation becomes in phase with the wave amplitude. It is shown that for low viscosity liquids the phase angle between the pressure perturbation exerted by the gas phase on the liquid at the gas–liquid interface and the wave amplitude, which is the measure of the relative effectiveness of the 'lift' and wave-drag components of the pressure perturbation, is a function of the density ratio (ratio of gas density at throat conditions to liquid density). At low density ratios both of these components are operative; however, at high density ratios the wave-drag component becomes dominant. The analysis further shows that the cut-off wavenumber and the wavenumber at maximum instability decrease with increasing density ratio. For highly viscous liquids and liquids having finite viscosity the pressure perturbation is always out of phase with the wave amplitude, and no cut-off wavenumber exists, i.e. the gas–liquid interface is always unstable in spite of the stabilizing effect of viscosity and surface tension.

1. Introduction

Several investigators have examined analytically the Kelvin–Helmholtz instability of a gas–liquid interface. Chang & Russell (1965) analysed the case of a plane liquid layer exposed to subsonic and supersonic gas streams. Nachtsheim

(1970) has considered the three-dimensional disturbance of a shear flow of a thin liquid film adjacent to a supersonic gas stream with wave fronts oblique to the external stream. A nonlinear analysis of the Kelvin–Helmholtz instability of a liquid film adjacent to a compressible gas and under the influence of a body force directed either away from or towards the liquid has been presented by Nayfeh & Saric (1971). Drazin (1970) has analysed the nonlinear Kelvin–Helmholtz instability of two parallel horizontal streams of inviscid incompressible fluid. Miles (1957, 1959, 1962) considered the effect of the critical layer present in the parallel incompressible shear flow of gas over a slightly viscous liquid performing two-dimensional wave motions. Benjamin (1959) extended the analysis of Miles (1957) by including the effect of the gas viscosity, and determined the pressure and shear stresses exerted by an incompressible stream on a rigid wavy wall. Craik (1966) used Benjamin's results to analyse the effect of the shear and pressure perturbations exerted by an incompressible gas stream on the stability of a liquid film.

In the present investigation, the stability of the gas–liquid interface of a sonic gas jet submerged in an infinite mass of liquid under the action of a pressure perturbation, liquid viscosity and surface tension is considered.

The presence of a high-speed flow of compressible gas over a liquid layer causes perturbations in interfacial stresses owing to the appearance of waves, and the Kelvin–Helmholtz instability is predominantly governed by the pressure perturbation exerted by the gas on the interface (Chang & Russell 1965). The analyses (Chang & Russell 1965; Nachsheim 1970; Nayfeh & Saric 1971) performed to date, at least to first order, used a linearized compressible flow theory neglecting the transient motion of the gas for subsonic and supersonic gas flows, and cannot be extended to include transonic gas flow, for their results would predict an infinite pressure perturbation at a Mach number $M_0 = 1$. This limitation is due to the neglect of terms due to the transient motion of the gas and the nonlinear term in the governing equation due to slow accumulation of disturbances near $M_0 \sim 1$ in the basic steady flow; the necessity for retaining the latter term will depend on the degree of unsteadiness. The neglect of transient gas motion in the case of subsonic and supersonic gas flows is a perfectly valid assumption provided, of course, that the wave velocity of the disturbance at the gas–liquid interface is much less than the gas velocity. Such an assumption becomes invalid for transonic or sonic flow, even for relatively slow oscillations of the disturbed gas–liquid interface. It may further be emphasized that for very slow oscillations the small perturbation equation for transonic flow may be nonlinear. If, however, there is a high enough rate of time variation, i.e. the unsteadiness introduced into the flow by oscillation of the gas–liquid interface is sufficiently large, the nonlinear disturbance accumulation does not have time to develop and a linearized treatment that includes the transient motion becomes justified. The present analysis includes an order-of-magnitude analysis of the full governing equations for the gas flow in an axisymmetric sonic gas jet submerged in a liquid with disturbances at the gas–liquid interface, and obtains the condition under which a linearized treatment is applicable. The method of linearizing the equations of motion is an extension of a method originally developed for flow past a two-dimensional oscillating wing by Lin, Reissner & Tsien (1948). It is demonstrated in this paper that sonic-gas-jet/liquid systems likely to be of physical interest satisfy this condition.

The basic difference in the destabilizing effect of the three regimes of gas flow, viz. subsonic (including incompressible), supersonic and sonic, on liquid layers is in the manner in which a pressure perturbation acts on the interface. The pressure perturbation exerted on the interface by subsonic gas flowing over a liquid surface is out of phase with the surface amplitude by 180°, and thus acts on the interface by pushing at the troughs of the wave and sucking at the crests (Nayfeh & Saric 1971). For supersonic gas flowing over the liquid surface, the pressure perturbation is in phase with the wave slope (Nayfeh & Saric 1971; Liepmann & Roshko 1957, p. 213); for sonic flow, however, the pressure perturbation is out of phase with the wave amplitude by an angle which varies with flow and instability parameters. For supersonic flow, the energy is transferred to the liquid layer predominantly through wave drag (Nayfeh & Saric 1971); for sonic flow, depending upon the value of the phase angle, energy may be transferred to the liquid layer by both the 'lift' and drag components of the pressure perturbation. At low liquid viscosities, the behaviour of the disturbed liquid layer at the interface for sonic flow is somewhat similar to that for subsonic flow in that each has a definite cut-off wavenumber above which the disturbance on the liquid layer is stable. At high liquid viscosities, the behaviour of the disturbed liquid layer for sonic flow is unlike that for subsonic and supersonic gas flows in that the liquid layer is always unstable; in subsonic flow, the cut-off wavenumber is unaltered by viscosity, and for supersonic flow the liquid layer is always stable.

The phase of the gas-pressure perturbation (as predicted by potential flow theory) with respect to disturbance waves at the gas–liquid interface can be altered by the Mach-number profile, as was demonstrated by Inger (1971, 1972). In his study of steady disturbances to the mean flow in a compressible boundary layer flowing past a slightly wave-swept wall, he found that, for a supersonic external flow, the Mach-number profile in the boundary layer has a marked effect on the phase angle of the wall pressure. However, his analysis showed no effect of the Mach-number profile on the phase of the wall pressure with respect to the wave amplitude for subsonic and transonic flows outside the boundary layer. For unsteady disturbances having a finite wave velocity in a boundary layer (compressible or incompressible) over a wavy surface, the phase of the wall pressure can also be shifted by the presence of a critical layer near the liquid layer. The effect of the critical layer in shifting the phase of the pressure perturbation exerted by the gas phase on the liquid layer at the gas–liquid interface for subsonic and supersonic flows outside the boundary layer is expected to be small, because the wave velocity is very small in comparison with the uniform gas velocity outside the boundary layer and the critical layer lies almost at the gas–liquid interface. In fact, the motion of the gas phase in these boundary layers is well represented by a quasi-steady approximation with a negligible contribution from the transient motion (see, for example, Bordner, Nayfeh & Saric 1973). Although Miles (1957, 1959, 1962) treated the case of an incompressible boundary layer, there does not appear to be any analysis available in the literature that

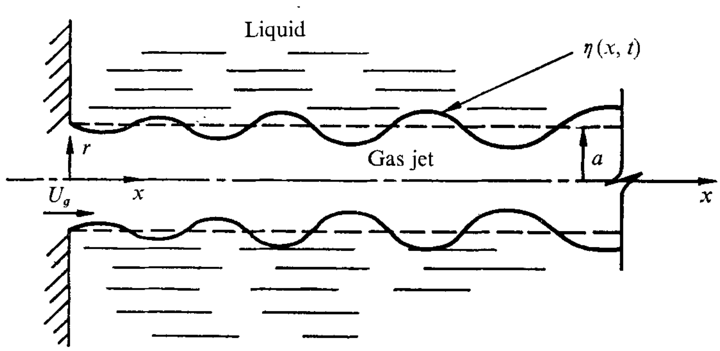

FIGURE 1. Description of the co-ordinate system.

treats the effect of the critical layer for the case of unsteady disturbances in a compressible boundary layer. The analysis presented here will, however, provide the outer matching condition for the distribution of the pressure perturbation in the boundary layer for the case of unsteady transonic flow outside the boundary layer.

2. Governing equations for gas jet

The physical problem considered is shown in figure 1. The gas jet issues from an orifice of radius a. The flow at the orifice is assumed uniform, and for a sonic jet has a velocity equal to the sonic velocity at the throat conditions. The expansion of the mean jet boundary is not considered, in anticipation of the short-wave approximation to be employed in the subsequent analysis, which reduces the axisymmetric two-dimensional gas–liquid system to a two-dimensional planar configuration. Therefore, it is assumed that gas jet has a constant mean radius equal to the orifice radius. Furthermore, we assume a continuous, frictionless (also neglecting the thin shear layer at the jet boundary), non-heat-conducting gas flow free from shock waves of finite strength and body forces. The basis for neglecting the shear layer at the jet boundary is that the pressure force, which dominates at the liquid surface compared with the friction force, is not affected by the presence of a shear layer (Chang & Russell 1965). It is also assumed that this shear layer does not significantly alter the phase of the gas-pressure perturbation with respect to the disturbance wave. The absence of body forces ensures that the stagnation enthalpy is constant; the assumption of a frictionless non-heat-conducting gas flow free from shock waves of finite strength ensures that the flow is isentropic, and we may assume the existence of a velocity potential ϕ_g which satisfies the nonlinear wave equation

$$\frac{\partial^2\phi_g}{\partial t^2}+2\frac{\partial\phi_g}{\partial x}\frac{\partial^2\phi_g}{\partial x\,\partial t}+2\frac{\partial\phi_g}{\partial r}\frac{\partial^2\phi_g}{\partial r\,\partial t}+\left(\frac{\partial\phi_g}{\partial x}\right)^2\frac{\partial^2\phi_g}{\partial x^2}$$
$$+2\frac{\partial\phi_g}{\partial x}\frac{\partial\phi_g}{\partial r}\frac{\partial^2\phi_g}{\partial x\,\partial r}+\left(\frac{\partial\phi_g}{\partial r}\right)^2\frac{\partial^2\phi_g}{\partial r^2}=c^2\left(\frac{\partial^2\phi_g}{\partial x^2}+\frac{\partial^2\phi_g}{\partial r^2}+\frac{1}{r}\frac{\partial\phi_g}{\partial r}\right), \quad (2.1)$$

where t is the time and c, the sonic velocity at a point in the jet, is given by

$$\frac{c^2}{\gamma-1}+\frac{\partial\phi_g}{\partial t}+\frac{1}{2}\left[\left(\frac{\partial\phi_g}{\partial x}\right)^2+\left(\frac{\partial\phi_g}{\partial r}\right)^2\right] = \tfrac{1}{2}U_g^2+\frac{c_0^2}{\gamma-1}. \tag{2.2}$$

Here U_g and c_0 are the velocities of the gas and sound at the orifice, i.e. at $x = 0$, respectively (for a sonic gas jet $U_g = c_0$), and γ is the ratio of specific heats. These governing equations are very general and are applicable to all regimes of axisymmetric compressible potential gas flow; in the next section these equations will be linearized and specialized to each of the three regimes of gas flow, viz. subsonic, sonic and supersonic. The solutions of (2.1) and (2.2) must satisfy the following kinematic boundary condition at the gas–liquid interface:

$$\frac{d\eta}{dt} = \frac{\partial\eta}{\partial t}+\frac{\partial\eta}{\partial x}\frac{dx}{dt}, \tag{2.3}$$

where η is the displacement of the gas–liquid interface from the mean jet radius.

3. Linearization of equations of motion for the gas phase

Let us introduce the following dimensionless variables:

$$x^* = x/\lambda, \quad y^* = (a-r)\,\delta/a, \quad t^* = \omega^*(tU_g/\lambda), \tag{3.1}$$

as well as a dimensionless perturbation potential ϕ and displacement η^* such that

$$\phi_g(r,x,t) = U_g x+\epsilon_\phi U_g\lambda\phi(r^*,x^*,t^*), \tag{3.2}$$

$$\eta(x,t) = \epsilon a\eta^*(x^*,t^*). \tag{3.3}$$

The dimensionless parameters ϵ_ϕ, ϵ, ω^* and δ are to be defined in a given neighbourhood in such a way that ϕ, η^* and their derivatives with respect to any of the dimensionless independent variables are $O(1)$. In particular,

$$\omega^* = |\alpha|\,\lambda/2\pi U_g, \tag{3.4}$$

where α is the (complex) angular frequency and λ the wavelength of the disturbance. If a linearized solution to the nonlinear equations is desired for short waves, i.e. for $\lambda/a \ll 1$, then we expand ϕ_g in a power series in λ/a, i.e. $\epsilon_\phi = \lambda/a$, and retain only the leading term in the expansion.

In terms of the above non-dimensional variables, the velocity components and the velocity of sound are

$$\frac{1}{U_g}\frac{\partial\phi_g}{\partial x} = 1+\epsilon_\phi\phi_{x^*}, \quad \frac{1}{U_g}\frac{\partial\phi_g}{\partial r} = -\epsilon_\phi\frac{\delta\lambda}{a}\phi_{y^*} \tag{3.5), (3.6}$$

and

$$\frac{c^2}{U_g^2} = \frac{1}{M_0^2}-(\gamma-1)\epsilon_\phi\left[\omega^*\phi_{t^*}+\phi_{x^*}+\tfrac{1}{2}\epsilon_\phi\left(\phi_{x^*}^2+\left(\frac{\delta\lambda}{a}\right)^2\phi_{y^*}^2\right)\right]. \tag{3.7}$$

The pressure coefficient is

$$C_p = \frac{P_g-P_0}{\frac{1}{2}\rho_g U_g^2} = \frac{2}{\gamma M_0^2}\left\{\left[1-(\gamma-1)\epsilon_\phi M_0^2\left(\omega^*\phi_{t^*}+\phi_{x^*}+\tfrac{1}{2}\epsilon_\phi\left(\phi_{x^*}^2+\left(\frac{\delta\lambda}{a}\right)^2\phi_{y^*}^2\right)\right)\right]^{\gamma/(\gamma-1)}-1\right\}. \tag{3.8}$$

Equation (2.1) becomes

$$(\omega^* a/\lambda)^2 \phi_{t^*t^*} + 2\omega^*(a/\lambda)^2 (1+\epsilon_\phi \phi_{x^*})\phi_{x^*t^*} + 2\epsilon_\phi \delta^2 \omega^* \phi_{y^*}\phi_{y^*t^*}$$
$$+ (a/\lambda)^2 (1+2\epsilon_\phi \phi_{x^*} + \epsilon_\phi^2 \phi_{x^*}^2)\phi_{x^*x^*} + 2\epsilon_\phi \delta^2 (1+\epsilon_\phi \phi_{x^*})\phi_{y^*}\phi_{x^*y^*}$$
$$+ \epsilon_\phi^2 \left(\frac{\delta^2\lambda}{a}\right)^2 \phi_{y^*}^2 \phi_{y^*y^*} = \left\{\frac{1}{M_0^2} - (\gamma-1)\epsilon_\phi \left[\omega^*\phi_{t^*} + \phi_{x^*}\right.\right.$$
$$\left.\left. + \tfrac{1}{2}\epsilon_\phi \left(\phi_{x^*}^2 + \left(\frac{\delta\lambda}{a}\right)^2 \phi_{y^*}^2\right)\right]\right\} \left[\left(\frac{a}{\lambda}\right)^2 \phi_{x^*x^*} + \delta^2 \phi_{y^*y^*} - \frac{\delta^2}{\delta - y^*}\phi_{y^*}\right], \quad (3.9)$$

and the boundary condition (2.3) becomes

$$-\delta\epsilon_\phi (\lambda/a)^2 \phi_{y^*} = \epsilon[\omega^* \eta_{t^*}^* + \eta_{x^*}^*(1+\epsilon_\phi \phi_{x^*})]. \quad (3.10)$$

In the above set of equations, P_g is the pressure at any point in the gas jet, while P_0, ρ_g and M_0 are the pressure, density and Mach number of the gas at the orifice. When specialized to a sonic jet, P_0 and ρ_g denote the critical values, i.e. values at $M_0 = 1$.

The theory of small perturbations requires that the deviations of the velocity components and pressure from the reference conditions be small. Thus, from (3.5)–(3.8), we obtain the following conditions:

$$\epsilon_\phi \ll 1, \quad \epsilon_\phi(\delta\lambda/a) \ll 1, \quad \epsilon_\phi M_0^2 \ll 1, \quad \epsilon_\phi M_0^2 \omega^* \ll 1, \quad (\epsilon_\phi M_0 \delta\lambda/a)^2 \ll 1. \quad (3.11)$$

With the use of the first of the above conditions, (3.10) simplifies to

$$-\delta\epsilon_\phi (\lambda/a)^2 \phi_{y^*} = \epsilon[\omega^* \eta_{t^*}^* + \eta_{x^*}^*], \quad (3.12)$$

and (3.9) to

$$\phi_{y^*y^*} - \frac{1}{\delta - y^*}\phi_{y^*} = (M_0\omega^*)^2 \frac{1}{\delta^2}\left(\frac{a}{\lambda}\right)^2 \phi_{t^*t^*} + 2M_0^2 \omega^* \frac{1}{\delta^2}\left(\frac{a}{\lambda}\right)^2 \phi_{x^*t^*}$$
$$+ 2\epsilon_\phi M_0^2 \phi_{y^*}\phi_{x^*y^*} + \frac{1}{\delta^2}\left(\frac{a}{\lambda}\right)^2 (M_0^2 - 1)\phi_{x^*x^*} + (\gamma+1) M_0^2 \frac{1}{\delta^2}\left(\frac{a}{\lambda}\right)^2 \epsilon_\phi \phi_{x^*}\phi_{x^*x^*}$$
$$+ 2\omega^* \epsilon_\phi M_0^2 \phi_{y^*}\phi_{y^*t^*} + (\gamma-1) M_0^2 (a/\delta\lambda)^2 \epsilon_\phi \omega^* \phi_{*}\phi_{x^*x^*}. \quad (3.13)$$

In the above equations, the orders of magnitude of the different terms may not necessarily be the same. However, all terms in (3.9) and (3.10) that are neglected are definitely small compared with terms retained in (3.12) and (3.13). Any further simplification depends on the order of magnitude of the coefficients of the retained terms.

Equation (3.12) implies that the larger of the two coefficients $\epsilon\omega^*$ and ϵ must be of order $\delta\epsilon_\phi(\lambda/a)^2$; the particular choice depends on whether

(i) $\omega^* \ll 1$, (ii) $\omega^* = O(1)$, (iii) $\omega^* \gg 1$.

In cases (i) and (ii) $$\delta\epsilon_\phi(\lambda/a)^2 = \epsilon, \quad (3.14)$$

and in case (iii) $$\delta\epsilon_\phi(\lambda/a)^2 = \epsilon\omega^*. \quad (3.15)$$

However, as will be seen subsequently, only cases (i) and (ii) are of physical relevance to the present problem. Physically, case (i) implies that the wave velocity is much less than the gas velocity and case (ii) implies that the wave velocity is of the same order as the gas velocity. Case (iii), which corresponds to highly transient motion, will not be considered here.

For case (i) with $M_0 = 1$, (3.13) further simplifies to

$$\phi_{y^*y^*} - \frac{1}{\delta(1-y^*/\delta)}\phi_{y^*} = (\gamma+1)\phi_{x^*}\phi_{x^*x^*} + 2\frac{\omega^*}{\epsilon_\phi}\phi_{x^*t^*}, \tag{3.16}$$

where in (3.13) we have set

$$\frac{1}{\delta^2}\left(\frac{a}{\lambda}\right)^2 \epsilon_\phi = 1, \quad \omega^* = O(\epsilon_\phi) \ll 1. \tag{3.17a}$$

For short waves, we obtain from (3.14) and (3.17*a*)

$$\epsilon_\phi = \lambda/a, \quad \delta = (a/\lambda)^{\frac{1}{2}}, \quad \epsilon = (\lambda/a)^{\frac{3}{2}}, \quad \omega^* = O(\lambda/a) \ll 1. \tag{3.17b}$$

Clearly, from the second of conditions (3.17*b*), it follows that $\delta \gg 1$, and consequently with $\phi_{y^*} = O(1)$ and $y^* = O(1)$, (3.16) reduces to the equation for a two-dimensional planar gas flow. These arguments also apply to other short-wave approximations for the various cases discussed below; therefore, for the sake of conciseness, they will not be repeated for each and every case considered in the following analysis. However, if $\omega^* \gg \epsilon_\phi$ but $\omega^* \ll 1$ [case (i)], it is clear that the transient motion will dominate and the nonlinear term in (3.16) becomes of higher order. Then (3.13) simplifies further to

$$\phi_{y^*y^*} - \frac{1}{\delta(1-y^*/\delta)}\phi_{y^*} = 2\phi_{x^*t^*}, \tag{3.18}$$

where in (3.13) the choice for the coefficients now becomes

$$\frac{\omega^*}{\delta^2}\left(\frac{a}{\lambda}\right)^2 = 1, \quad \omega^* \gg \epsilon_\phi, \quad \omega^* \ll 1. \tag{3.19a}$$

The above conditions when specialized to short waves become

$$\epsilon_\phi = \lambda/a, \quad \delta = \omega^{*\frac{1}{2}}(a/\lambda), \quad \epsilon = \omega^{*\frac{1}{2}}(\lambda/a)^2, \quad \omega^* \gg \lambda/a, \quad \omega^* \ll 1. \tag{3.19b}$$

For subsonic ($M_0 < 1$) and supersonic ($M_0 > 1$) flows in case (i), (3.13) simplifies to

$$\phi_{y^*y^*} - \frac{1}{\delta(1-y^*/\delta)}\phi_{y^*} = \frac{M_0^2-1}{|M_0^2-1|}\phi_{x^*x^*}, \tag{3.20}$$

where in (3.13) we have set

$$\frac{|M_0^2-1|}{\delta^2}\left(\frac{a}{\lambda}\right)^2 = 1, \quad \epsilon_\phi \ll |1-M_0^{-2}|, \quad \omega^* \ll |1-M_0^{-2}|, \tag{3.21a}$$

or, for short waves,

$$\epsilon_\phi = \lambda/a \ll |1-M_0^{-2}|, \quad \delta = |M_0^2-1|^{\frac{1}{2}}(a/\lambda),$$
$$\epsilon = (\lambda/a)^2|M_0-1|^{\frac{1}{2}}, \quad \omega^* \ll |1-M_0^{-2}|. \tag{3.21b}$$

It is clear from the first of conditions (3.21*b*) that for short waves, i.e. $\lambda/a \ll 1$, the condition for the applicability of the linearized equation (3.20) for subsonic and supersonic (not including $M_0 \to 1$) gas jets can easily be satisfied.

For case (i), i.e. $\omega^* \ll 1$, with the use of (3.14), equations (3.12) and (3.8) for all three regimes (subsonic, transonic and supersonic) simplify to, respectively,

$$-\phi_{y^*} = \eta^*_{x^*}, \quad C_p = -2\epsilon_\phi\phi_{x^*}. \qquad (3.22), (3.23)$$

From the foregoing analysis, it is clear that, so long as $\omega^* \ll 1$ (i.e. the wave velocity is much less than the gas velocity), the transient motion of the gas can be neglected for subsonic and supersonic flows; however, for sonic flow, the transient motion of the gas is dominant to the extent that if $\omega^* = O(\epsilon_\phi = \lambda/a)$ the contribution of transient variation of the axial component of motion becomes of the same order as the nonlinear disturbance accumulation term. However, if $\omega^* \gg \epsilon_\phi = \lambda/a$ and still $\omega^* \ll 1$, the nonlinear disturbance accumulation does not have time to develop; the above-mentioned transient component of motion of the gas dominates and the governing equation becomes linear.

It is self-evident from the above analysis and discussion that for case (ii) the transient motion of the gas will predominate. As a result, (3.13), (3.12) and (3.8) for $M_0 = 1$ simplify to, respectively,

$$\phi_{y^*y^*} - \frac{1}{\delta(1-y^*/\delta)}\phi_{y^*} = \omega^*\phi_{t^*t^*} + 2\phi_{x^*t^*}, \tag{3.24}$$

$$-\phi_{y^*} = \omega^*\eta^*_{t^*} + \eta^*_{x^*}, \tag{3.25}$$

$$C_p = -2\epsilon_\phi(\omega^*\phi_{t^*} + \phi_{x^*}), \tag{3.26}$$

where in (3.13) we have made the choice

$$\omega^* = O(1), \quad \frac{\omega^*}{\delta^2}\left(\frac{a}{\lambda}\right)^2 = 1, \quad \omega^* \gg \epsilon_\phi. \tag{3.27a}$$

In obtaining (3.25) and (3.26), conditions (3.27a) have been used. For short waves, conditions (3.27a) become

$$\omega^* = O(1), \quad \omega^* \gg \epsilon_\phi = \lambda/a, \quad \delta = \omega^{*\frac{1}{2}}(a/\lambda), \quad \epsilon = \omega^{*\frac{1}{2}}(\lambda/a)^2. \tag{3.27b}$$

The corresponding governing equations for case (ii) for subsonic and supersonic flows can easily be deduced. However, almost all gas–liquid systems of physical interest satisfy conditions (3.21). The analysis of the stability problem associated with subsonic and supersonic gas flows adjacent to a plane liquid layer for case (i) has already been presented in Chang & Russell (1965) and Nayfeh & Saric (1971); therefore, in what follows we restrict our attention to sonic flow only. It will be demonstrated in the subsequent analysis that almost all sonic-gas-jet/liquid systems of physical interest satisfy either the conditions (3.19b) or (3.27b) for application of the short-wave approximation, depending on the values of the flow parameters. Since case (ii) embraces case (i), and the planar form of (3.18), (3.22) and (3.23) can be deduced from the planar form of (3.24)–(3.26) by letting $\omega^* \ll 1$, we shall therefore, in the present analysis, use the following dimensional form of these latter equations reduced to a planar gas flow by use of the third of conditions (3.27b):

$$\frac{\partial^2\phi_g}{\partial r^2} - \frac{2}{U_g}\frac{\partial^2\phi_g}{\partial t\,\partial x} - \frac{1}{U_g^2}\frac{\partial^2\phi_g}{\partial t^2} = 0, \tag{3.28}$$

$$\frac{\partial\phi_g}{\partial r} = \frac{\partial\eta}{\partial t} + U_g\frac{\partial\eta}{\partial x} \quad \text{at} \quad r = a, \tag{3.29}$$

$$P_g = \frac{\gamma+1}{\gamma}\rho_g U_g^2 - \rho_g\left[\frac{\partial\phi_g}{\partial t} + U_g\frac{\partial\phi_g}{\partial x}\right]. \tag{3.30}$$

4. Equations of motion for the liquid phase

The motion of the liquid surrounding the gas jet is assumed to be described by the following linearized equations consistent with the short-wave approximation discussed previously:

$$\partial u/\partial x + \partial v/\partial r = 0, \tag{4.1}$$

$$\frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial r^2}\right], \tag{4.2}$$

$$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial P}{\partial r} + \nu\left[\frac{\partial v^2}{\partial x^2} + \frac{\partial^2 v}{\partial r^2}\right], \tag{4.3}$$

where u and v are the axial and radial components of the induced motion of the liquid, respectively, P is the pressure at any point in the liquid, ν is the kinematic viscosity of the liquid and ρ is its density.

The linearized kinematic boundary condition at the gas–liquid interface that solutions to the above equations must satisfy is

$$v = \partial\eta/\partial t. \tag{4.4}$$

In addition, these solutions must also satisfy the following dynamic boundary conditions. With the neglect of the shear stress exerted by the gas on the gas–liquid interface, the continuity of tangential stresses at the interface requires that

$$\mu(\partial u/\partial r + \partial v/\partial x) = 0, \tag{4.5}$$

where μ is the dynamic viscosity of the liquid. The continuity of normal stresses at the gas–liquid interface requires that

$$-P + 2\mu\frac{\partial v}{\partial r} + P_g = \frac{\sigma}{a} - \sigma\frac{\partial^2\eta}{\partial x^2}, \tag{4.6}$$

where σ is the surface tension. The above equation is consistent with the short-wave approximation discussed previously.

Equations (4.2) and (4.3) can be simplified by use of a stream function $\psi(r, x, t)$ defined by

$$u = \partial\psi/\partial r, \quad v = -\partial\psi/\partial x, \tag{4.7}$$

which follows directly from the continuity equation (4.1). The substitution of the above into (4.2) and (4.3) yields after eliminating the pressure P between the resulting equations

$$(\nabla^2\psi - \nu^{-1}\,\partial/\partial t)\,\nabla^2\psi = 0, \tag{4.8}$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial r^2$. Since the operators $\nabla^2 - \nu^{-1}\,\partial/\partial t$ and ∇^2 commute, the function ψ can be separated into two parts: ψ_1, which satisfies

$$\nabla^2\psi_1 = 0, \tag{4.9}$$

and ψ_2, which satisfies

$$\nabla^2\psi_2 - \nu^{-1}\,\partial\psi_2/\partial t = 0. \tag{4.10}$$

The solutions of (4.9) and (4.10) can then be combined to give the general solution of (4.8) as

$$\psi = \psi_1 + \psi_2. \tag{4.11}$$

Substituting (4.7) and (4.11) into (4.2) and (4.3), and using (4.9) and (4.10), we obtain

$$\frac{1}{\rho}\frac{\partial P}{\partial x} = -\frac{\partial}{\partial t}\left(\frac{\partial \psi_1}{\partial r}\right), \quad \frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{\partial}{\partial t}\left(\frac{\partial \psi_1}{\partial x}\right). \tag{4.12), (4.13}$$

5. Solutions

As is usual in linear stability analysis, we assume that the gas–liquid interface is set into motion according to the expression

$$\eta(x,t) = \eta_0 e^{i\kappa x + \alpha t}, \tag{5.1}$$

where κ is the wavenumber and η_0 is the amplitude. In view of (5.1), the solution of (3.28) that vanishes at the jet axis and satisfies boundary condition (3.29) at the gas–liquid interface is

$$\phi_g(r,x,t) = U_g x + (\alpha + i\kappa U_g)\frac{\sin(\kappa_c r)}{\kappa_c \cos(\kappa_c a)}\eta. \tag{5.2}$$

The solutions of (4.9) and (4.10) which vanish at infinity and when combined in the manner of (4.11) satisfy (4.4) and (4.5) at the gas–liquid interface are, respectively,

$$\psi_1 = -\frac{i\alpha(l^2+\kappa^2)}{\kappa(\kappa^2 - l^2)}\eta\, e^{-\kappa(r-a)}, \tag{5.3a}$$

$$\psi_2 = \frac{2i\alpha\kappa}{\kappa^2 - l^2}\eta\, e^{-l(r-a)}. \tag{5.3b}$$

We have introduced in (5.2)

$$\kappa_c^2 = -\kappa^2\left[\frac{2i\alpha}{U_g\kappa} + \left(\frac{\alpha}{U_g\kappa}\right)^2\right], \tag{5.4}$$

and in (5.3)

$$l^2 = \kappa^2 + \alpha/\nu. \tag{5.5}$$

The use of (5.2) in (3.30) yields

$$\Delta P_g = P_g - \frac{1}{\gamma}\rho_g U_g^2 = -\rho_g(\alpha + i\kappa U_g)^2 \frac{\sin(\kappa_c r)}{\kappa_c \cos(\kappa_c a)}\eta. \tag{5.6}$$

Equation (5.6) for the gas-pressure perturbation can be simplified considerably when applied at $r = a$: with α expressed as

$$\alpha = \alpha_r - i\alpha_i \tag{5.7}$$

(where α_r is the time amplification factor and α_i is the angular frequency of the disturbance), we obtain from (5.4)

$$\kappa_c = \kappa_{cr} - i\kappa_{ci}, \tag{5.8a}$$

where

$$\kappa_{cr} = \left(\frac{\kappa}{U_g}\right)^{\frac{1}{2}}\left\{\left[\left(\alpha_i + \frac{\alpha_r^2 - \alpha_i^2}{2U_g\kappa}\right)^2 + \left(\alpha_r - \frac{\alpha_i\alpha_r}{U_g\kappa}\right)^2\right]^{\frac{1}{2}} - \left(\alpha_i + \frac{\alpha_r^2 - \alpha_i^2}{2U_g\kappa}\right)\right\}^{\frac{1}{2}}, \tag{5.8b}$$

$$\kappa_{ci} = \left(\frac{\kappa}{U_g}\right)^{\frac{1}{2}}\left\{\left[\left(\alpha_i + \frac{\alpha_r^2 - \alpha_i^2}{2U_g\kappa}\right)^2 + \left(\alpha_r - \frac{\alpha_i\alpha_r}{U_g\kappa}\right)^2\right]^{\frac{1}{2}} + \left(\alpha_i + \frac{\alpha_r^2 - \alpha_i^2}{2U_g\kappa}\right)\right\}^{\frac{1}{2}}. \tag{5.8c}$$

Clearly,

$$-\mathrm{Im}(\kappa_c) \equiv \kappa_{ci} > \mathrm{Re}(\kappa_c) \equiv \kappa_{cr} > 0;$$

further, if $a|\kappa_c| \gg 1$, then

$$a\kappa_{ci} \gg 1. \tag{5.9}$$

Substituting (5.8) into the expressions

$$\cos(\kappa_c a) = \frac{e^{i\kappa_c a} + e^{-i\kappa_c a}}{2}, \quad \sin(\kappa_c a) = \frac{e^{i\kappa_c a} - e^{-i\kappa_c a}}{2i} \tag{5.10}$$

and using (5.9), we obtain the following simple asymptotic formulae for $\cos(\kappa_c a)$ and $\sin(\kappa_c a)$:

$$\cos(\kappa_c a) \sim \tfrac{1}{2} e^{i\kappa_c a}, \quad \sin(\kappa_c a) \sim (2i)^{-1} e^{i\kappa_c a}. \tag{5.11}$$

Using them in (5.6) at $r = a$ yields

$$\Delta P_g = \rho_g(\alpha + i\kappa U_g^2)\,\eta/\kappa_c. \tag{5.12}$$

To obtain the phase angle of the pressure perturbation acting at the gas–liquid interface, we take the real part of (5.12). Use of (5.7) and (5.8a) then yields

$$\begin{aligned}\operatorname{Re}[\Delta P_g] &= \operatorname{Re}[\rho_g(\alpha + i\kappa U_g)^2 \eta/\kappa_c] \\ &= (\rho_g U_g^2 \kappa^2/|\kappa_c|^2)\,[(w_r\kappa_{ci} - w_i\kappa_{cr})^2 + (w_r\kappa_{cr} + w_i\kappa_{ci})^2]^{\frac{1}{2}} \\ &\qquad \times \eta_0 \cos[\kappa x - \alpha_i t - (\tfrac{1}{2}\pi - \theta)]\, e^{\alpha_r t}, \end{aligned} \tag{5.13}$$

where

$$w_r = 1 - \frac{2\alpha_i}{\kappa U_g} + \left(\frac{\alpha_i}{\kappa U_g}\right)^2 - \left(\frac{\alpha_r}{\kappa U_g}\right)^2, \quad w_i = 2\left[\frac{\alpha_r}{\kappa U_g} - \frac{\alpha_r\alpha_i}{(\kappa U_g)^2}\right], \tag{5.14}$$

$$|\kappa_c|^2 = \kappa_{cr}^2 + \kappa_{ci}^2. \tag{5.15}$$

Equation (5.13) shows that at the gas–liquid interface the pressure perturbation is out of phase with the wave amplitude by an angle $\frac{1}{2}\pi - \theta$, where the angle θ is given by

$$\theta = \tan^{-1}[(w_r\kappa_{ci} - w_i\kappa_{cr})/(w_r\kappa_{cr} + w_i\kappa_{ci})]. \tag{5.16a}$$

For case (i), i.e. with the use of $|\alpha|/\kappa U_g \ll 1$ in the expressions (5.8b, c) for κ_{cr} and κ_{ci} and in the expressions (5.14) for w_r and w_i, we obtain

$$\theta = \tan^{-1}\{[(\alpha_i^2 + \alpha_r^2)^{\frac{1}{2}} + \alpha_i]^{\frac{1}{2}}/[(\alpha_i^2 + \alpha_r^2)^{\frac{1}{2}} - \alpha_i]^{\frac{1}{2}}\}. \tag{5.16b}$$

It may be noted that θ is a measure of the relative effectiveness of the wave-drag component as compared with the 'lift' component of the pressure perturbation. For example, with $\theta = 90°$, the pressure perturbation is in phase with the wave amplitude; as a result, energy is transferred to the liquid layer at the gas–liquid interface predominantly through the lift component, as in subsonic gas flow. With $\theta = 0$, the pressure perturbation is in antiphase with the wave slope; as a result, the dominant mode of energy transfer is through wave drag, as in supersonic gas flow.

The expression for the pressure P in the liquid phase is obtained by substituting (5.3a) into (4.12) or (4.13) and integrating; the result is

$$P = C - \rho\alpha^2 \frac{\kappa^2 + l^2}{\kappa(\kappa^2 - l^2)}\,\eta\, e^{-\kappa(r-a)}, \tag{5.17}$$

where C is a constant and is evaluated by applying (4.6) at $x = 0$ (i.e. at the location where the disturbance originates; at $x = 0$, the deviation η of the gas–liquid interface is zero). Thus,

$$C = \gamma^{-1}\rho_g U_g^2 - \sigma/a. \tag{5.18}$$

Finally, with the constant C given by the above expression, and by substitution of (5.17), (5.12), (5.3) and (5.1) into (4.6), we obtain at $r = a$

$$\frac{\kappa^2+l^2}{\kappa^2-l^2}\alpha^2 + 2\nu\kappa^2\alpha\frac{(\kappa-l)^2}{\kappa^2-l^2} + i\frac{\rho_g\kappa}{\rho\kappa_c}(i\kappa U_g+\alpha)^2 = \frac{\sigma\kappa^3}{\rho}. \tag{5.19}$$

6. Solution of the dispersion equation

For ease in obtaining roots, (5.19) is separated into real and imaginary parts by substituting for l, α and κ_c from (5.5), (5.7) and (5.8), respectively. The resultant equations in dimensionless form are

$$-4\Gamma_\mu\kappa^{*2}\alpha_r^* - \alpha_r^{*2} + \alpha_i^{*2} - 4\Gamma_\mu^2\kappa^{*4} + 8^{\frac{1}{2}}\Gamma_\mu^2\kappa^{*4}l_r^* + (\Gamma_\mu\kappa^{*3}/|\kappa_c^*|^2)(w_r^*\kappa_{ci}^* - w_i^*\kappa_{cr}^*) = \Gamma_\mu\kappa^{*3} \tag{6.1a}$$

and

$$4\Gamma_\mu\kappa^{*2}\alpha_i^* + 2\alpha_r^*\alpha_i^* - 8^{\frac{1}{2}}\Gamma_\mu^2\kappa^{*4}l_i^* - (\Gamma_\mu\kappa^{*3}/|\kappa_c^*|^2)(w_r^*\kappa_{cr}^* + w_i^*\kappa_{ci}^*) = 0, \tag{6.1b}$$

where

$$\kappa^* = \frac{\kappa\sigma^{\frac{3}{2}}}{\mu^{\frac{1}{2}}\rho_g U_g^{\frac{5}{2}}},\quad \alpha_r^* = \frac{\alpha_r(\mu\sigma)^{\frac{1}{2}}}{\rho_g U_g^{\frac{3}{2}}},\quad \alpha_i^* = \frac{\alpha_i(\mu\sigma)^{\frac{1}{2}}}{\rho_g U_g^{\frac{3}{2}}},\quad \Gamma_\mu = \frac{\rho_g}{\rho}\left(\frac{\mu U_g}{\sigma}\right)^{\frac{5}{2}}, \tag{6.2}$$

$$l_r^* = \left\{\left[\left(1+\frac{\alpha_r^*}{\Gamma_\mu\kappa^{*2}}\right)^2 + \left(\frac{\alpha_i^*}{\Gamma_\mu\kappa^{*2}}\right)^2\right]^{\frac{1}{2}} + \left(1+\frac{\alpha_r^*}{\Gamma_\mu\kappa^{*2}}\right)\right\}^{\frac{1}{2}}, \tag{6.3a}$$

$$l_i^* = \left\{\left[\left(1+\frac{\alpha_r^*}{\Gamma_\mu\kappa^{*2}}\right)^2 + \left(\frac{\alpha_i^*}{\Gamma_\mu\kappa^{*2}}\right)^2\right]^{\frac{1}{2}} - \left(1+\frac{\alpha_r^*}{\Gamma_\mu\kappa^{*2}}\right)\right\}^{\frac{1}{2}}, \tag{6.3b}$$

$$w_r^* = w_r = 1 - 2\Gamma_\omega\frac{\alpha_i^*}{\kappa^*} + \left(\frac{\Gamma_\omega\alpha_i^*}{\kappa^*}\right)^2 - \left(\frac{\Gamma_\omega\alpha_r^*}{\kappa^*}\right)^2, \tag{6.4a}$$

$$w_i^* = w_i = 2\frac{\Gamma_\omega\alpha_r^*}{\kappa^*}\left(1-\frac{\Gamma_\omega\alpha_i^*}{\kappa^*}\right), \tag{6.4b}$$

$$\kappa_{cr}^* = \kappa^{*\frac{1}{2}}\left\{\left[\left(\alpha_i^*+\Gamma_\omega\frac{\alpha_r^{*2}-\alpha_i^{*2}}{2\kappa^*}\right)^2 + \left(\alpha_r^* - \Gamma_\omega\frac{\alpha_r^*\alpha_i^*}{\kappa^*}\right)^2\right]^{\frac{1}{2}} - \left(\alpha_i^*+\Gamma_\omega\frac{\alpha_r^{*2}-\alpha_i^{*2}}{2\kappa^*}\right)\right\}^{\frac{1}{2}}, \tag{6.5a}$$

$$\kappa_{ci} = \kappa^{*\frac{1}{2}}\left\{\left[\left(\alpha_i^*+\Gamma_\omega\frac{\alpha_r^{*2}-\alpha_i^{*2}}{2\kappa^*}\right)^2 + \left(\alpha_r^* - \Gamma_\omega\frac{\alpha_r^*\alpha_i^*}{\kappa^*}\right)^2\right]^{\frac{1}{2}} + \left(\alpha_i^*+\Gamma_\omega\frac{\alpha_r^{*2}-\alpha_i^{*2}}{2\kappa^*}\right)\right\}^{\frac{1}{2}}, \tag{6.5b}$$

$$|\kappa_c^*|^2 = \kappa_{cr}^{*2} + \kappa_{ci}^{*2},\quad \Gamma_\omega = \sigma/\mu U_g = (\rho_g/\rho\Gamma_\mu)^{\frac{2}{5}}. \tag{6.6a,b}$$

The reason for the above choice for the combination of flow parameters in order to make α_r, α_i and κ dimensionless will be made clear in the subsequent analysis.

The solution of the above equations consists of obtaining

$$\alpha_r^* = \alpha_r^*(\kappa^*, \Gamma_\mu, \Gamma_\omega),\quad \alpha_i^* = \alpha_i^*(\kappa^*, \Gamma_\mu, \Gamma_\omega), \tag{6.7}$$

where, for a given value of Γ_μ, (6.6*b*) shows that Γ_ω is only a function of the density ratio ρ_g/ρ. Thus, in the solution of (6.1) for α_r^* and α_i^* as functions of κ^*, the parameters are Γ_μ and ρ_g/ρ. However, prior to obtaining such a solution it will be very instructive to obtain the general form of the solutions; this consists of seeking solutions of (6.1) for low and high values of the liquid viscosity with and without the limitation imposed by the condition of case (i), and searching for maxima of α_r^* (if they exist) with respect to κ^* for various values of Γ_μ and Γ_ω or the density ratio.

6.1. *Approximation for a low viscosity liquid*

A low viscosity liquid is defined as one for which the following condition holds:

$$|\alpha| \gg \nu\kappa^2, \quad \text{i.e.} \quad \Gamma_\mu \kappa^{*2}/|\alpha^*| \ll 1. \tag{6.8}$$

The use of the above condition in (6.1) yields

$$\alpha_i^{*2} - \alpha_r^{*2} + (\Gamma_\mu \kappa^{*3}/|\kappa_c^*|^2)(w_r^* \kappa_{ci}^* - w_i^* \kappa_{cr}^*) = \Gamma_\mu \kappa^{*3}, \tag{6.9}$$

$$2\alpha_r^* \alpha_i^* - (\Gamma_\mu \kappa^{*3}/|\kappa_c^*|^2)(w_r^* \kappa_{cr}^* + w_i^* \kappa_{ci}^*) = 0. \tag{6.10}$$

To show that the above equations are indeed independent of the viscosity of liquid, we make the substitutions

$$\hat{\alpha}_r = \alpha_r^* \Gamma_\mu^{-0\cdot 2}, \quad \hat{\alpha}_i = \alpha_i^* \Gamma_\mu^{-0\cdot 2}, \quad \hat{\kappa} = \kappa^* \Gamma_\mu^{0\cdot 2} \tag{6.11}$$

in (6.9) and (6.10), obtaining

$$\hat{\alpha}_i^2 - \hat{\alpha}_r^2 + (\hat{\kappa}^3/|\hat{\kappa}_c|^2)(\hat{w}_r \hat{\kappa}_{ci} - \hat{w}_i \hat{\kappa}_{cr}) = \hat{\kappa}^3, \tag{6.12a}$$

$$2\hat{\alpha}_i - \frac{\hat{\kappa}^3}{|\hat{\kappa}_c|^2}\left\{\left[\hat{\kappa} - \hat{\alpha}_i\left(\frac{\rho_g}{\rho}\right)^{0\cdot 4}\right]\frac{\hat{w}_r}{\hat{\kappa}_{ci}} + 2\left(\frac{\rho_g}{\rho}\right)^{0\cdot 4}\left[1 - \left(\frac{\rho_g}{\rho}\right)^{0\cdot 4}\frac{\hat{\alpha}_i}{\hat{\kappa}}\right]\frac{\hat{\kappa}_{ci}}{\hat{\kappa}}\right\} = 0, \tag{6.12b}$$

where

$$\hat{w}_r = 1 - 2\left(\frac{\rho_g}{\rho}\right)^{0\cdot 4}\frac{\hat{\alpha}_i}{\hat{\kappa}} + \left(\frac{\rho_g}{\rho}\right)^{0\cdot 8}\left(\frac{\hat{\alpha}_r}{\hat{\kappa}}\right)^2 - \left(\frac{\rho_g}{\rho}\right)^{0\cdot 8}\left(\frac{\hat{\alpha}_r}{\hat{\kappa}}\right)^2, \tag{6.13a}$$

$$\hat{w}_i = 2\left(\frac{\rho_g}{\rho}\right)^{0\cdot 4}\frac{\hat{\alpha}_r}{\hat{\kappa}}\left[1 - \left(\frac{\rho_g}{\rho}\right)^{0\cdot 4}\frac{\hat{\alpha}_i}{\hat{\kappa}}\right], \tag{6.13b}$$

$$\hat{\kappa}_{cr} = \hat{\kappa}^{\frac{1}{2}}\left\{\left[\left(\hat{\alpha}_i + \left(\frac{\rho_g}{\rho}\right)^{0\cdot 4}\frac{\hat{\alpha}_r^2 - \hat{\alpha}_i^2}{2\hat{\kappa}}\right)^2 + \left(\hat{\alpha}_r - \left(\frac{\rho_g}{\rho}\right)^{0\cdot 4}\frac{\hat{\alpha}_r\hat{\alpha}_i}{\hat{\kappa}}\right)^2\right]^{\frac{1}{2}} - \left[\hat{\alpha}_i + \left(\frac{\rho_g}{\rho}\right)^{0\cdot 4}\frac{\hat{\alpha}_r^2 - \hat{\alpha}_i^2}{2\hat{\kappa}}\right]\right\}^{\frac{1}{2}}, \tag{6.14a}$$

$$\hat{\kappa}_{ci} = \hat{\kappa}^{\frac{1}{2}}\left\{\left[\left(\hat{\alpha}_i + \left(\frac{\rho_g}{\rho}\right)^{0\cdot 4}\frac{\hat{\alpha}_r^2 - \hat{\alpha}_i^2}{2\hat{\kappa}}\right)^2 + \left(\hat{\alpha}_r - \left(\frac{\rho_g}{\rho}\right)^{0\cdot 4}\frac{\hat{\alpha}_r\hat{\alpha}_i}{\hat{\kappa}}\right)^2\right]^{\frac{1}{2}} + \left[\hat{\alpha}_i + \left(\frac{\rho_g}{\rho}\right)^{0\cdot 4}\frac{\hat{\alpha}_r^2 - \hat{\alpha}_i^2}{2\hat{\kappa}}\right]\right\}^{\frac{1}{2}}, \tag{6.14b}$$

$$|\hat{\kappa}_c|^2 = \hat{\kappa}_{cr}^2 + \hat{\kappa}_{ci}^2. \tag{6.14c}$$

The above equations are clearly independent of the liquid viscosity. Note that, in obtaining (6.12*b*) from (6.10), the root $\hat{\alpha}_r = 0$ of (6.10) has been excluded. Thus, the stable behaviour of the gas–liquid interface is extracted by letting $\hat{\alpha}_r = 0$ in (6.12*a*); the resulting governing equation is

$$\hat{\alpha}_i^2 + \hat{\kappa}^3\left[1 - 2\left(\frac{\rho_g}{\rho}\right)^{0\cdot 4}\frac{\hat{\alpha}_i}{\hat{\kappa}} + \left(\frac{\rho_g}{\rho}\right)^{0\cdot 8}\left(\frac{\hat{\alpha}_i}{\hat{\kappa}}\right)^2\right]\Big/\left\{2\hat{\kappa}\hat{\alpha}_i\left[1 - \left(\frac{\rho_g}{\rho}\right)^{0\cdot 4}\frac{\hat{\alpha}_i}{2\hat{\kappa}}\right]\right\}^{\frac{1}{2}} = \hat{\kappa}^3. \tag{6.15}$$

Equations (6.12)–(6.14) clearly show that the solution of (6.12) for $\hat{\alpha}_r$ and $\hat{\alpha}_i$ as functions of $\hat{\kappa}$ depends only on the parameter ρ_g/ρ, i.e.

$$\hat{\alpha}_r = \hat{\alpha}_r(\hat{\kappa}, \rho_g/\rho), \quad \hat{\alpha}_i = \hat{\alpha}_i(\hat{\kappa}, \rho_g/\rho). \tag{6.16}$$

Among the above solutions for various values of the parameter ρ_g/ρ, the one with $\rho_g/\rho \to 0$ corresponds to the case of low wave velocity, namely, case (i), as will be shown below. The condition for low wave velocity is

$$\frac{|\alpha|}{\kappa U_g} \ll 1 \quad \text{or} \quad \left(\frac{\rho_g}{\rho}\right)^{0\cdot4} \frac{|\hat{\alpha}|}{\hat{\kappa}} \ll 1. \tag{6.17}$$

The application of the above condition to (6.12) yields for an unstable gas–liquid interface

$$\hat{\alpha}_i^2 - \hat{\alpha}_r^2 + \frac{\hat{\kappa}^{\frac{5}{2}}}{2}\left(\frac{(\hat{\alpha}_i^2+\hat{\alpha}_r^2)^{\frac{1}{2}}+\hat{\alpha}_i}{\hat{\alpha}_i^2+\hat{\alpha}_r^2}\right)^{\frac{1}{2}} = \hat{\kappa}^3, \tag{6.18a}$$

$$2\hat{\alpha}_i - \frac{\hat{\kappa}^{\frac{5}{2}}}{2\{(\hat{\alpha}_i^2+\hat{\alpha}_r^2)\,[(\hat{\alpha}_i^2+\hat{\alpha}_r^2)^{\frac{1}{2}}+\hat{\alpha}_i]\}^{\frac{1}{2}}} = 0. \tag{6.18b}$$

For a stable gas–liquid interface, we obtain from (6.15)

$$\hat{\alpha}_i^2 + \tfrac{1}{2}\hat{\kappa}^{\frac{5}{2}}(2/\hat{\alpha}_i)^{\frac{1}{2}} = \hat{\kappa}^3. \tag{6.19}$$

Thus, for the low wave velocities (6.12) and (6.15) become independent of the density-ratio parameter.

The solutions of (6.18b) and (6.19) give the following values for the neutrally stable (cut-off) wavenumber and the corresponding frequency:

$$\hat{\kappa} = \tfrac{5}{4}, \quad \hat{\alpha}_i = \tfrac{5}{8}. \tag{6.20}$$

For a stable mode of disturbance (5.16b) gives $\theta = 90°$, i.e. the pressure perturbation given by (5.13) becomes in phase with the wave amplitude. As a result, the local wave-drag component of the pressure perturbation becomes small in comparison with the lift component.

The solution of (6.18) corresponding to the mode of maximum instability is obtained by differentiating (6.18) with respect to $\hat{\kappa}$, setting $d\hat{\alpha}_r/d\hat{\kappa} = 0$ and eliminating $d\hat{\alpha}_i/d\hat{\kappa}$ between the resulting equations; the third equation thus obtained is solved simultaneously with (6.18), yielding

$$\hat{\alpha}_{rm} = \frac{\alpha_{rm}\sigma}{\rho_g U_g^3}\left(\frac{\rho}{\rho_g}\right)^{0\cdot2} \simeq 0\cdot357, \quad \hat{\alpha}_{im} = \frac{\alpha_{im}\sigma}{\rho_g U_g^3}\left(\frac{\rho}{\rho_g}\right)^{0\cdot2} \simeq 0\cdot329, \tag{6.21a}$$

$$\hat{\kappa}_m = \frac{\kappa_m \sigma}{\rho_g U_g^2}\left(\frac{\rho_g}{\rho}\right)^{0\cdot2} \simeq 0\cdot803, \tag{6.21b}$$

where the subscript m has been added to denote the values of $\hat{\kappa}$ and $\hat{\alpha}_i$ at the maximum value $\hat{\alpha}_{rm}$ of the amplification factor of the waves. These equations show that the surface tension acts to reduce the frequency and rate of amplification at maximum instability according to an inverse law for σ. With the above solution, conditions (6.8) and (6.17) become, respectively,

$$\Gamma_\mu^{0\cdot4} \ll 1, \quad (\rho_g/\rho)^{0\cdot4} \ll 1, \tag{6.22a, b}$$

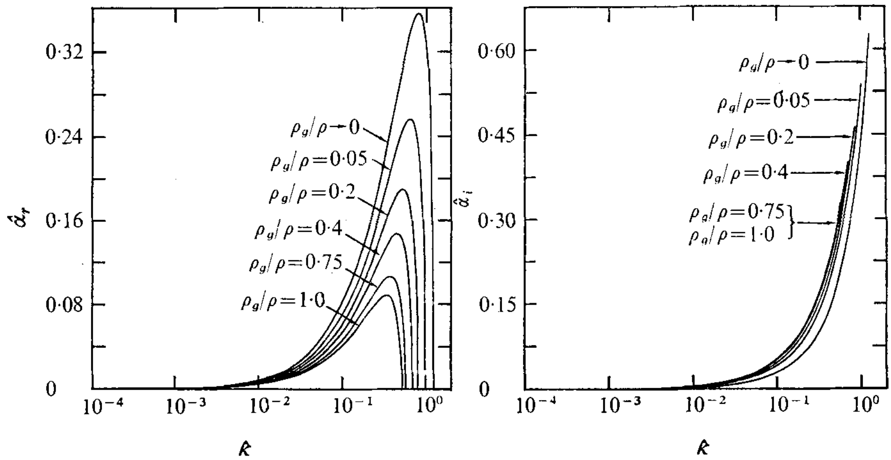

FIGURE 2. The dependence of the non-dimensional amplification factor and frequency on the non-dimensional wavenumber and on the density ratio ρ_g/ρ for low viscosity liquids.

and the condition of linearization, namely,

$$\omega^* \gg \lambda/a \quad \text{or} \quad \frac{\Gamma_\omega|\alpha^*|}{\kappa^*} = \left(\frac{\rho_g}{\rho}\right)^{0\cdot 4}\frac{|\hat{\alpha}|}{\hat{k}} \gg \lambda/a, \tag{6.23a}$$

becomes

$$a \gg 13\left(\frac{\rho}{\rho_g}\right)^{0\cdot 2}\frac{\sigma}{\rho_g U_g^2} \quad \text{or} \quad We = \frac{2a\rho_g U_g^2}{\sigma} \gg 26\left(\frac{\rho}{\rho_g}\right)^{0\cdot 2}, \tag{6.23b}$$

where We is the Weber number based on the nozzle diameter. The above condition clearly shows that the linearization of the wave equation becomes more valid as the gas density or the stagnation pressure increases. In addition, the conditions necessary for employing the short-wave approximation, viz.,

$$\kappa a \gg 1, \quad |l|a \gg 1, \quad |\kappa_c|a \gg 1, \tag{6.24a}$$

must also be satisfied. From consideration of (5.4), (5.5) and condition (6.17) it is clear that only the third of the above inequalities needs to be satisfied. From (5.8) and (6.21) and condition (6.17), the third of inequalities (6.24a) becomes

$$a \gg \sigma/\rho_g U_g^2. \tag{6.24b}$$

From comparison of condition (6.23b) with condition (6.24b), it is evident that the former is more stringent than the latter; therefore, condition (6.24b) becomes redundant.

The solution of (6.18) as a function of wavenumber is obtained below as a special case of a more general solution as indicated by the functional form (6.16). For this purpose, (6.12) are solved numerically, varying ρ_g/ρ from 0 to 1·0 to cover a very wide range of stagnation pressures or gas–liquid systems. Figure 2 displays such solutions of (6.12). It can be seen that in the inviscid case there exists one neutrally stable (or cut-off) wavenumber, which varies with the density ratio or wave velocity. At or near the cut-off wavenumber, the rate of amplification decays very rapidly owing to a very rapid decrease in wave drag (i.e. θ

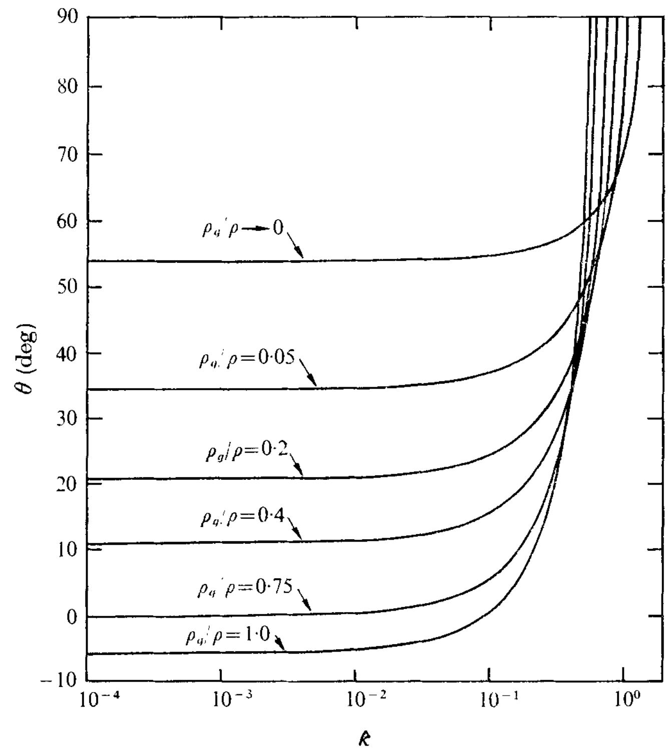

FIGURE 3. Angle θ as a function of the non-dimensional wavenumber with the density ratio ρ_g/ρ as a parameter for low viscosity liquids.

approaching 90°; see figure 3), and the disturbance above the cut-off wavenumber consists of pure oscillatory motion with constant amplitude. Figure 3, which displays θ as a function of wavenumber, shows that at low density ratios and for values of the wavenumber below the value at maximum instability, the energy is transferred to the liquid layer at the interface owing to both wave drag and 'lift'; however, with increasing density ratio, the wave-drag component of the pressure perturbation increases in comparison with the 'lift' component and becomes most effective (i.e. $\theta = 0$) at $\rho_g/\rho \simeq 0{\cdot}75$. Further, at maximum instability the wave-drag component of the pressure perturbation increases monotonically in comparison with the 'lift' component with increasing density ratio, as can be seen in figure 3. The manner in which the cut-off wavenumber and wavenumber at maximum instability vary with density ratio can be discerned from figure 2. It can be seen that the cut-off wavenumber and the wavenumber of the maximum instability decrease with increasing density ratio, i.e. the modes of stability and maximum instability shift towards lower wavenumbers with increasing density ratios.

For supersonic gas flow (Chang & Russell 1965), the dominant mode of energy transfer from the gas phase to the liquid layer at the gas–liquid interface is by means of wave drag; however, unlike the case of sonic gas flow, for supersonic gas flow and low viscosity liquids there does not exist any cut-off wavenumber, and the condition at the gas–liquid interface is always one of instability in spite of the stabilizing effect of surface tension. However, with respect to the density ratio, the rate of amplification in the case of supersonic flow (Chang & Russell 1965) displays a behaviour similar to that in the sonic case [see (6.21)] in the vicinity of low wave velocities, i.e. decreases with decreasing density ratio, becoming zero in the limit of $\rho_g/\rho = 0$.

With subsonic gas flow (Nayfeh & Saric 1971) over a liquid surface, energy is imparted to the liquid layer at the gas–liquid interface by the pressure perturbation in the gas phase, which pushes out at the wave troughs and sucks in at the wave crests; as in the sonic case, there exists a cut-off wavenumber owing to the stabilizing effect of surface tension. Above the cut-off wavenumber, the disturbance is stable and reduces to a pure oscillatory motion with constant amplitude. In the unstable mode of disturbance, subsonic flow, like sonic flow, displays a maximum with respect to wavenumber; in contrast to the sonic case the frequency of the disturbance motion is always zero and the cut-off wavenumber is independent of the density ratio. As for sonic and supersonic gas flows, in subsonic flow (Chang & Russell 1965) at low wave velocities the rate of amplification also decreases with decreasing density ratio.

6.2. *Approximation for a high viscosity liquid*

A highly viscous liquid is characterized by the condition

$$|\alpha| \ll \nu\kappa^2, \quad \text{i.e.} \quad |\alpha^*|/\Gamma_\mu \kappa^{*2} \ll 1. \tag{6.25}$$

The application of this condition to (6.1) yields

$$-2\alpha_r^* + (\kappa^*/|\kappa_c^*|^2)(w_r^*\kappa_{ci}^* - w_i^*\kappa_{cr}^*) = \kappa^* \tag{6.26a}$$

and

$$-2\alpha_i^* - (\kappa^*/|\kappa_c^*|^2)(w_r^*\kappa_{cr}^* + w_i^*\kappa_{ci}^*) = 0. \tag{6.26b}$$

The free parameter that must be varied in order to obtain a solution of the above equations is Γ_ω, i.e. the functional form of the solution is

$$\alpha_r^* = \alpha_r^*(\kappa^*, \Gamma_\omega), \quad \alpha_i^* = \alpha_i^*(\kappa^*, \Gamma_\omega). \tag{6.27}$$

In analogy with the case of low viscosity liquids, $\Gamma_\omega \to 0$ corresponds to the case of low wave velocity. To appreciate better the nature of the general solution in the form given in (6.27), we study below the case of low wave velocity.

The condition (6.17) for the approximation of low wave velocity in terms of the non-dimensional variables defined by (6.2) becomes

$$\Gamma_\omega |\alpha^*|/\kappa^* \ll 1. \tag{6.28}$$

The use of this condition in (6.26) yields

$$-2\alpha_r^* + \frac{\kappa^{*\frac{1}{2}}}{2}\left(\frac{(\alpha_i^{*2}+\alpha_r^{*2})^{\frac{1}{2}}+\alpha_i^*}{\alpha_i^{*2}+\alpha_r^{*2}}\right)^{\frac{1}{2}} = \kappa^* \tag{6.29a}$$

and

$$2\alpha_i^* - \frac{\kappa^{*\frac{1}{2}}}{2}\left(\frac{(\alpha_i^{*2}+\alpha_r^{*2})^{\frac{1}{2}}-\alpha_i^*}{\alpha_i^{*2}+\alpha_r^{*2}}\right)^{\frac{1}{2}} = 0. \tag{6.29b}$$

Clearly the above equations are independent of Γ_ω. From them it can be seen that there does not exist any stable mode of disturbance. The solution corre-

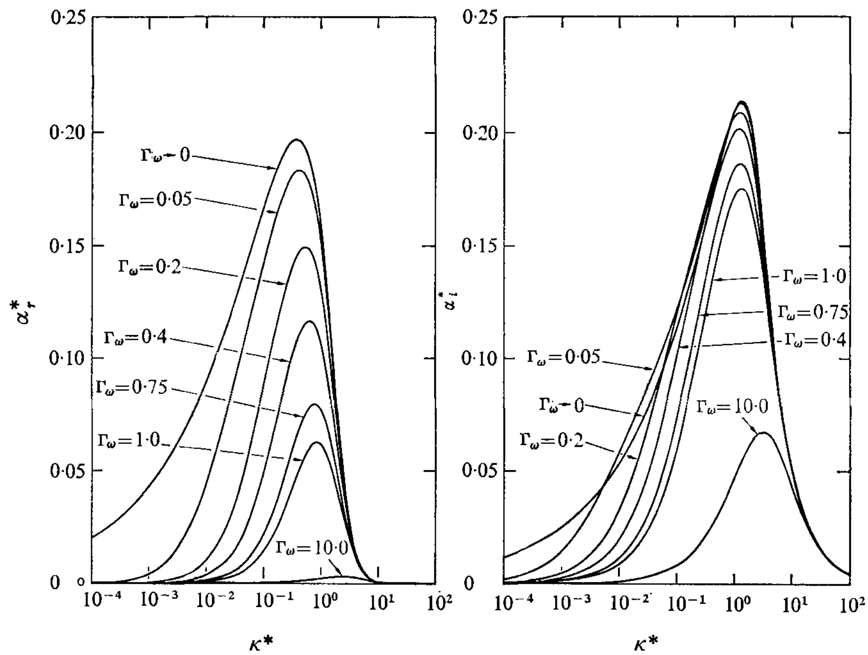

FIGURE 4. The variation of the non-dimensional amplification factor and frequency as functions of the non-dimensional wavenumber with Γ_ω as a parameter for high viscosity liquids.

sponding to the mode of maximum instability is obtained in a way similar to that for low viscosity liquids satisfying the condition of low wave velocity and is given by

$$\alpha_{rm}^* = \frac{\alpha_{rm}}{\rho_g U_g^{\frac{5}{2}}/(\mu\sigma)^{\frac{1}{2}}} \simeq 0·197, \quad \alpha_{im}^* = \frac{\alpha_{im}}{\rho_g U_g^{\frac{5}{2}}/(\mu\sigma)^{\frac{1}{2}}} \simeq 0·171; \tag{6.30a}$$

$$\kappa_m^* = \frac{\kappa_m}{\rho_g U_g^{\frac{3}{2}} \mu^{\frac{1}{2}}/\sigma^{\frac{3}{2}}} \simeq 0·352. \tag{6.30b}$$

It is the above set of equations that provided the motivation for employing the above combination of flow parameters for the purpose of non-dimensionalizing α_r, α_i and κ in the manner indicated by (6.2). The above solution shows that the viscosity and surface tension act to reduce the frequency and the rate of amplification at the maximum instability according to an inverse-square-root law for $\mu\sigma$.

With the solution given by (6.30) for α_r^*, α_i^* and κ^* at maximum instability, the conditions (6.25), (6.28) and (6.23*a*) become, respectively,

$$\Gamma_\mu \gg 1, \tag{6.31}$$

$$\Gamma_\omega \ll 1 \quad \text{or} \quad (\rho_g/\rho)^{0·4} \ll \Gamma_\mu^{0·4}, \tag{6.32}$$

$$\Gamma_\omega \gg \lambda/a \quad \text{or} \quad a \gg 18\sigma/(\Gamma_\omega^{\frac{1}{2}} \rho_g U_g^2) \quad \text{or} \quad We \gg 36/\Gamma_\omega^{\frac{1}{2}}. \tag{6.33}$$

Condition (6.33) clearly shows that, as the liquid viscosity increases, the condition for linearization is satisfied with less margin (see (6.6) for definition of Γ_ω). For the solution given by (6.30), once again, it can easily be shown that the third of

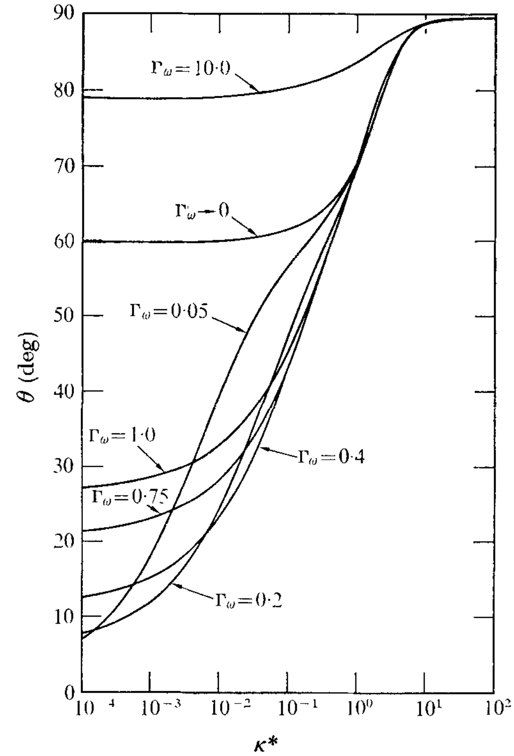

FIGURE 5. The variation of the angle θ as a function of the non-dimensional wavenumber with Γ_ω as a parameter for high viscosity liquids.

the conditions (6.24*a*) is less stringent than condition (6.33) and therefore in the case of high viscosity liquids also is no longer a necessary condition.

The solution of (6.26) as a function of wavenumber in the form of (6.27) with Γ_ω as a parameter is displayed in figures 4 and 5. Figure 4 shows that in highly viscous liquids, unlike low viscosity liquids, there does not exist any cut-off wavenumber, and the physical condition of the liquid layer at the interface is always one of instability owing to the destablizing effect of the 'lift' and wave-drag components of the pressure perturbation on the liquid layer. As can be seen from figure 5, which displays the angle θ as a function of the wavenumber κ^* with Γ_ω as a parameter, θ, unlike the case for low viscosity liquids, approaches 90° (i.e. the pressure perturbation becomes in phase with the wave amplitude) only asymptotically at large values of κ^*. In contrast to sonic gas flow, in subsonic gas flow (Chang & Russell 1965), the cut-off wavenumber is unaltered by the presence of viscosity, whereas for supersonic flow (Chang & Russell 1965) the liquid layer in the case of highly viscous liquids is always stable. Distributions of the amplification factor and the frequency as a function of wavenumber shown in figure 4 possess maxima, but these maxima occur at different wavenumbers. By comparing the plot of the amplification factor given in figure 4 with that given in figure 2 for low viscosity liquids, it can be seen that the effect of Γ_ω on the amplification factor near maximum instability is somewhat analogous to the

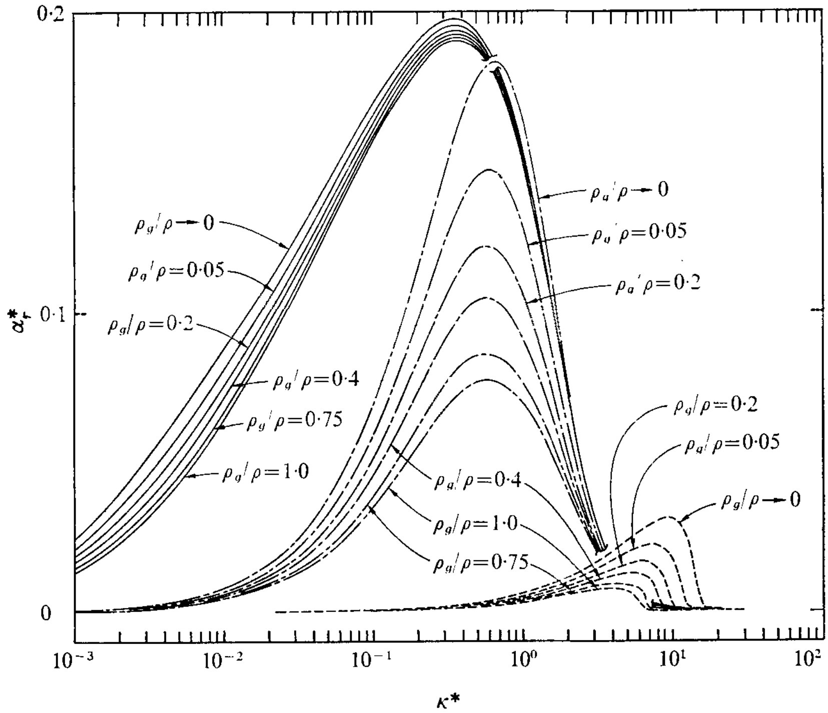

FIGURE 6. The variation of the non-dimensional amplification factor as a function of the non-dimensional wavenumber with Γ_μ and the density ratio ρ_g/ρ as parameters corresponding to the general solution of the dispersion equation. ——, $\Gamma_\mu = 10^4$; — — —, $\Gamma_\mu = 1{\cdot}0$; - - - -, $\Gamma_\mu = 5 \times 10^{-6}$.

effect of the density ratio in case of low viscosity liquids. It may be pointed out that, although we have considered a very wide range of values of Γ_ω for the case of highly viscous liquids, in practice it is not possible to realize these values for any physical system involving high viscosity liquids. For most physical systems values of Γ_ω lie well below $\Gamma_\omega = 0{\cdot}1$; this, in turn, implies that the approximation of low wave velocity may be a valid approximation for most highly viscous liquids.

6.3. *General solution*

The general solution of (6.1) obtained numerically in the form indicated by the functions (6.7) is presented in figures 6 and 7. The value of Γ_μ was varied over a wide range; however, for the sake of clarity in the graphical presentation, plots are shown only for three values of Γ_μ, corresponding, respectively, to very high, very low and intermediate values of Γ_μ. As discussed previously, for a given value of Γ_μ, the parameter Γ_ω depends only on the density ratio ρ_g/ρ; therefore, in these plots the density ratio ρ_g/ρ instead of Γ_ω is shown as a parameter. For very high values of Γ_μ, e.g. $\Gamma_\mu = 10^4$, the distributions of the amplification factor and frequency as functions of κ^* given in figures 6 and 7, respectively, are very similar to the corresponding distributions shown in figure 4 for high viscosity liquids. The reason for this similarity can be seen from the plot of $\Gamma_\mu \kappa^{*2}/|\alpha^*|$ as a function

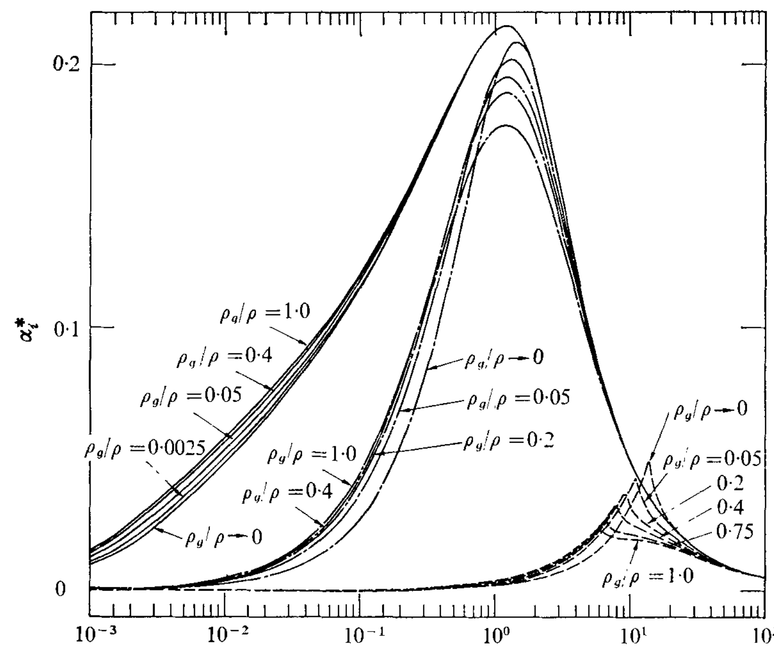

FIGURE 7. The variation of the non-dimensional frequency as a function of the non-dimensional wavenumber with Γ_μ and the density ratio ρ_g/ρ as parameters corresponding to the general solution of the dispersion equation. Curves as in figure 6.

of κ^* in figure 8; this figure shows that, for the value of Γ_μ given above, $\Gamma_\mu \kappa^{*2}/|\alpha^*|$ has a finite value which increases rapidly as κ^* increases and the condition for the approximation for a high viscosity liquid is satisfied with an increasing margin. Furthermore, it can be seen from figure 6 that, like the case for a high value of Γ_μ, there does not exist any cut-off wavenumber even at intermediate values of Γ_μ, e.g. $\Gamma_\mu = 1·0$, owing to the destablizing action of the 'lift' and wave-drag components of the pressure perturbation. As can be seen from figure 9, which shows the angle θ as a function of κ^* with Γ_μ and the density ratio as parameters, θ approaches 90° asymptotically at large values of κ^* for high and moderate values of Γ_μ. Figures 6 and 7 further show that, for high values of the wavenumber, the behaviour of both the frequency and amplification factor for intermediate values of Γ_μ approaches that for high values of Γ_μ; the explanation for this follows readily by reference to figure 8. For low values of Γ_μ, e.g. $\Gamma_\mu = 5 \times 10^{-6}$, the distributions of α_r^* and α_i^* behave very similarly to those at very low liquid viscosities so long as the condition $\Gamma_\mu \kappa^{*2}/|\alpha^*| \ll 1$ (see figure 8) is satisfied. This condition is satisfied, however, with a sufficiently wide margin for values of the wavenumber below those which correspond to maximum instability, and the margin decreases very rapidly as the wavenumber increases further, i.e. the effect of the viscosity of the liquid becomes more pronounced with increasing wavenumber. Consequently, the distribution of α_r^* does not show a well-defined cut-off wavenumber as in the approximation for a low viscosity

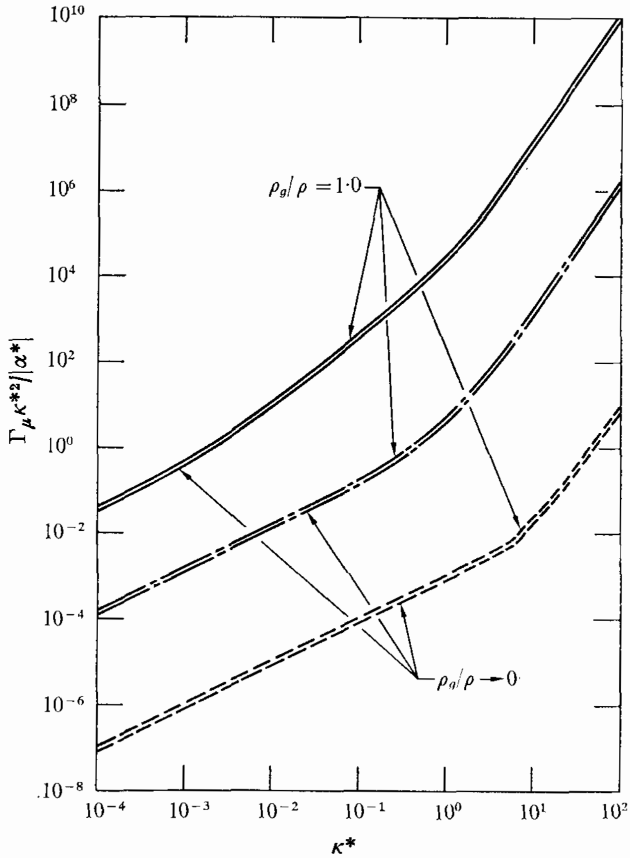

FIGURE 8. The variation of $\Gamma_\mu \kappa^{*2}/|\alpha^*|$ (parameter for liquid-viscosity condition) as a function of the non-dimensional wavenumber with Γ_μ and the density ratio ρ_g/ρ as parameters corresponding to the general solution of the dispersion equation. Curves as in figure 6.

liquid, although the rapid fall-off in the distribution of α_r^* and rapid increase in the angle θ, i.e. rapid decrease in the wave-drag component of the pressure perturbation (see figures 6 and 9) beyond the maximum instability as seen previously in figures 2 and 3 for low viscosity liquids owing to the stabilizing effect of surface tension, are also evident at very low values of Γ_μ.

7. Illustration of conditions for validity of various approximations

In order to illustrate the conditions for the approximations for low and high viscosity liquids with and without the approximation of low wave velocity, and the condition necessary for linearization of the governing wave equation for a sonic gas jet submerged in a liquid with a disturbance at the gas–liquid interface, we chose air and water as an example of the most commonly available substances, air and mercury as a typical example of gas–liquid systems in which the liquid is very heavy and has a very high surface tension, and air and glycerin as a typical

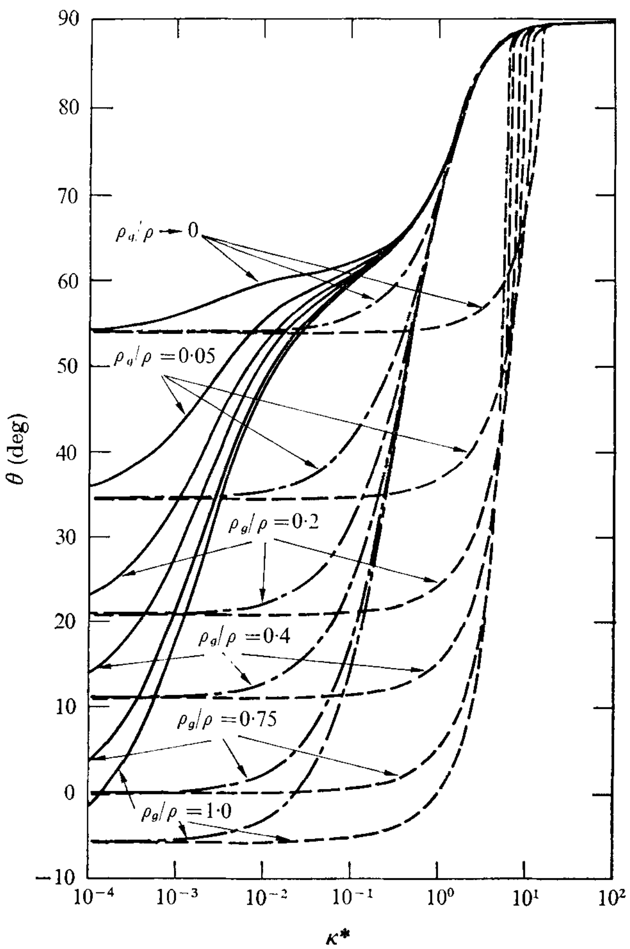

FIGURE 9. The variation of the angle θ as a function of the non-dimensional wavenumber with Γ_μ and the density ratio ρ_g/ρ as parameters corresponding to the general solution of the dispersion equation. Curves as in figure 6.

example of gas–liquid systems in which the liquid has a very high viscosity. Further, to illustrate the effect of changes in gas-flow parameters such as the velocity and gas density, we substituted xenon for air in these examples; xenon is a typical very heavy gas. To illustrate the above-mentioned conditions, we solved (6.1) and obtained the magnitude of these conditions at the maximum instability as a function of the density ratio. Figure 10 displays plot of $\lambda_m \kappa^*_m/\Gamma_\omega |\alpha^*_m|$ [see condition (6.23)] as a function of density ratio for the gas–liquid systems mentioned above. In each example, the temperature at the throat of jet was fixed at room temperature, viz. 20 °C, and the lowest value of the gas density at the throat chosen in each example corresponds to that at room temperature and pressure. Figure 10 shows that, for a given gas, fixed stagnation temperature and fixed density ratio, a jet of bigger diameter must be used for a highly viscous liquid than for a low viscosity liquid in order that the condition for linearization

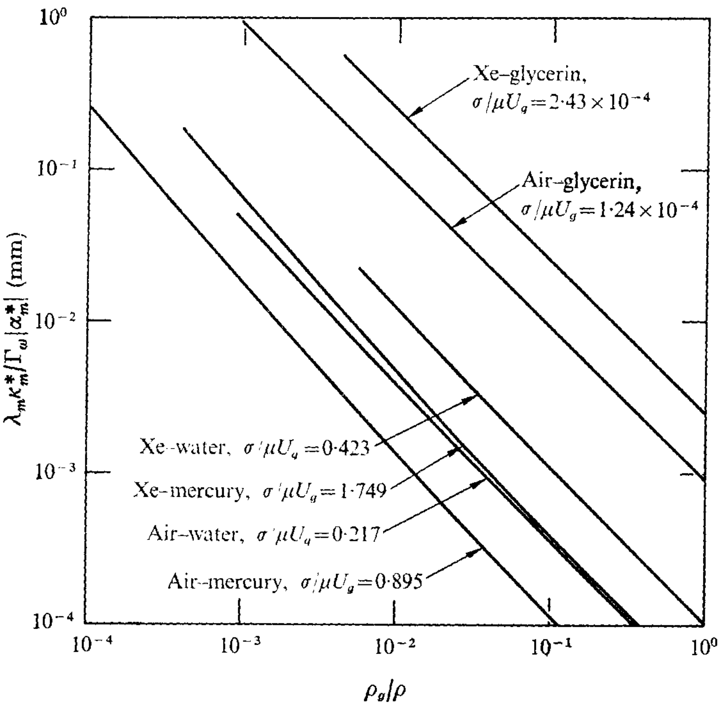

FIGURE 10. The variation of $\lambda_m \kappa_m^*/(\Gamma_\omega |\alpha_m^*|)$ corresponding to maximum instability as a function of the density ratio ρ_g/ρ for various gas–liquid systems.

of the wave equation be satisfied with the same margin. However, the values of the parameter $\lambda_m \kappa_m^* / \Gamma_\omega |\alpha_m^*|$ for the examples considered (which in the author's judgement embrace a very wide range of flow parameters) are so small that jet sizes together with the range of operating stagnation pressures employed in practice (e.g. in the field of pneumatic atomization) satisfy the condition for linearization of the wave equation, viz. $a \gg \lambda_m \kappa_m^* / (\Gamma_\omega |\alpha_m^*|)$. Furthermore, as figure 10 shows, the margin with which this condition is satisfied increases very rapidly with increasing density ratio, or for a given gas–liquid system, with increasing gas density. For high viscosity liquids [for which condition (6.25) or (6.31) applies] and low wave velocity [for which condition (6.28) or (6.32) applies] as typified by glycerin, figure 10 shows that the parameter $\lambda_m \kappa_m^* / \Gamma_\omega |\alpha_m^*|$ for a given gas–liquid system varies as $1/\rho_g$; this is also evident from condition (6.33). However, for low viscosity liquids and low wave velocity as typified approximately by the air–mercury system [see conditions (6.22)] at low density ratios, the parameter $\lambda_m \kappa_m^* / \Gamma_\omega |\alpha_m^*|$ varies approximately as $1/\rho_g^{1{\cdot}2}$ as is also evident from condition (6.23*b*).

8. Summary and conclusions

It is shown that the wave equation for a sonic gas jet submerged in a liquid with a disturbed gas–liquid interface can be linearized provided that the non-dimensional wave velocity $|\alpha|/\kappa U_g \gg \lambda/a$ or $|\alpha| a/U_g \gg 1$, i.e. that the time rate of perturbations predominates. This condition for linearization for low and high viscosity liquids and low wave velocity (i.e. $|\alpha|/\kappa U_g \ll 1$) reduces to, respectively, $We \gg 26(\rho/\rho_g)^{0\cdot 2}$ and $We \gg 36/\Gamma_\omega^{\frac{1}{2}}$. It is demonstrated that most gas–liquid systems of physical interest satisfy the condition for linearization of the wave equation; for highly viscous liquids, however, this condition is more stringent than for low viscosity liquids. The analysis shows that the pressure perturbation exerted by the gas phase on the liquid layer at the gas–liquid interface is out of phase with the wave amplitude by an angle which varies with the flow and instability parameters; consequently, energy is transferred to the liquid layer both through its wave-drag and 'lift' components. However, for low liquid viscosities the wave-drag component becomes very small in comparison with the lift component at and above the cut-off wavenumber. On the other hand, in the case of liquids having intermediate and large viscosities, the wave drag is always effective in the transfer of energy to the liquid layer, and there does not exist any cut-off wavenumber; the physical state of the liquid layer is always one of instability in spite of the stabilizing effect of surface tension and viscosity.

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